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Ellipse Through Four Points

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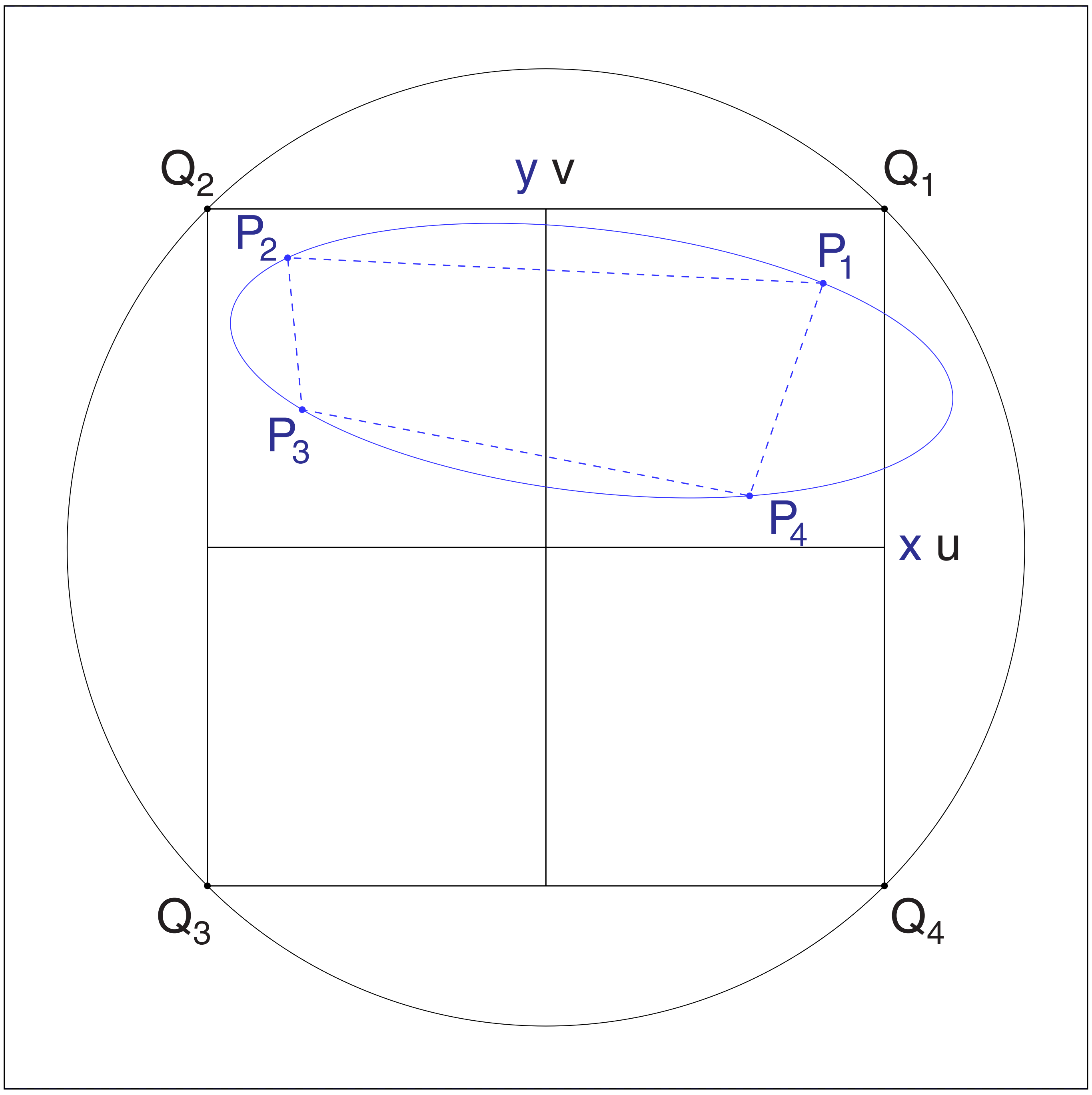
1. Introduction

Given are four points $P_1..P_4$ in xy-coordinates (blue). It is assumed that the convex hull of the four points is a quadrilateral and the points are sorted counterclockwise.
An ellipse should be drawn through these points. An ellipse is defined by five parameters (center x_0,y_0, half axes a,b and rotation angle), therefore four points are not sufficient.
The problem is solved by perspective mapping: the ellipse in xy is considered as the perspective image of a circle in uv-coordinates. $Q_1..Q_4$ are the corners ±1 of a square.
Once the functions x(u,v) and y(u,v) are found, the ellipse in xy can be drawn by mapping a sufficiently large number of points from the circle in uv to xy. The application on the title page is merely an illustration.



2.1 Mathematics / (u,v) to (x,y)

General perspective mapping is well-known. The formulas are valid for mapping from 3D to 2D (11 parameters) or for mapping from 2D to 2D (8 parameters) like here.

General perspective mapping / **u** to **x**

$$\mathbf{x} = (x,y)^T$$
$$\mathbf{u} = (u,v)^T$$
$$\mathbf{a} = (a_u,a_v)^T$$
$$\mathbf{b} = (b_u,b_v)^T$$
$$\mathbf{c} = (c_u,c_v)^T$$

$$x = \frac{a_0 + \mathbf{a}^T\mathbf{u}}{1+\mathbf{c}^T\mathbf{u}}$$

$$y = \frac{b_0 + \mathbf{b}^T\mathbf{u}}{1+\mathbf{c}^T\mathbf{u}}$$

$$a_0 + \mathbf{a}^T\mathbf{u} + 0 + 0 + 0 - x\,\mathbf{c}^T\mathbf{u} = x$$
$$0 + 0 + 0 + b_0 + \mathbf{b}^T\mathbf{u} - y\,\mathbf{c}^T\mathbf{u} = y$$

For four points and eight unknowns we get eight linear equations

$$\mathbf{p} = (a_0, a_u, a_v, b_0, b_u, b_v, c_u, c_v)^T$$
$$\mathbf{r} = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)^T$$

$$\mathbf{Ap} = \mathbf{r}$$

$$\mathbf{A} = \begin{bmatrix} 1 & u_1 & v_1 & 0 & 0 & 0 & -x_1u_1 & -x_1v_1 \\ 1 & u_2 & v_2 & 0 & 0 & 0 & -x_2u_2 & -x_2v_2 \\ 1 & u_3 & v_3 & 0 & 0 & 0 & -x_3u_3 & -x_3v_3 \\ 1 & u_4 & v_4 & 0 & 0 & 0 & -x_4u_4 & -x_4v_4 \\ 0 & 0 & 0 & 1 & u_1 & v_1 & -y_1u_1 & -y_1v_1 \\ 0 & 0 & 0 & 1 & u_2 & v_2 & -y_2u_2 & -y_2v_2 \\ 0 & 0 & 0 & 1 & u_3 & v_3 & -y_3u_3 & -y_3v_3 \\ 0 & 0 & 0 & 1 & u_4 & v_4 & -y_4u_4 & -y_4v_4 \end{bmatrix}$$

Special case regular square u,v

$$\mathbf{A} = \begin{bmatrix} +1 & +1 & +1 & 0 & 0 & 0 & -x_1 & -x_1 \\ +1 & -1 & +1 & 0 & 0 & 0 & +x_2 & -x_2 \\ +1 & -1 & -1 & 0 & 0 & 0 & +x_3 & +x_3 \\ +1 & +1 & -1 & 0 & 0 & 0 & -x_4 & +x_4 \\ 0 & 0 & 0 & +1 & +1 & +1 & -y_1 & -y_1 \\ 0 & 0 & 0 & +1 & -1 & +1 & +y_2 & -y_2 \\ 0 & 0 & 0 & +1 & -1 & -1 & +y_3 & +y_3 \\ 0 & 0 & 0 & +1 & +1 & -1 & -y_4 & +y_4 \end{bmatrix}$$

2.2 Mathematics / (x,y) to (u,v)

This is the inverse transform:

General perspective mapping / **x** to **u**

$$\mathbf{x} = (x,y)^T$$
$$\mathbf{u} = (u,v)^T$$
$$\mathbf{a} = (a_x,a_y)^T$$
$$\mathbf{b} = (b_x,b_y)^T$$
$$\mathbf{c} = (c_x,c_y)^T$$

$$u = \frac{a_0 + \mathbf{a}^T\mathbf{x}}{1+\mathbf{c}^T\mathbf{x}}$$

$$v = \frac{b_0 + \mathbf{b}^T\mathbf{x}}{1+\mathbf{c}^T\mathbf{x}}$$

$$a_0 + \mathbf{a}^T\mathbf{x} + 0 + 0 + 0 - u\mathbf{c}^T\mathbf{x} = u$$
$$0 + 0 + 0 + b_0 + \mathbf{b}^T\mathbf{x} - v\mathbf{c}^T\mathbf{x} = v$$

For four points and eight unknowns we get
eight linear equations

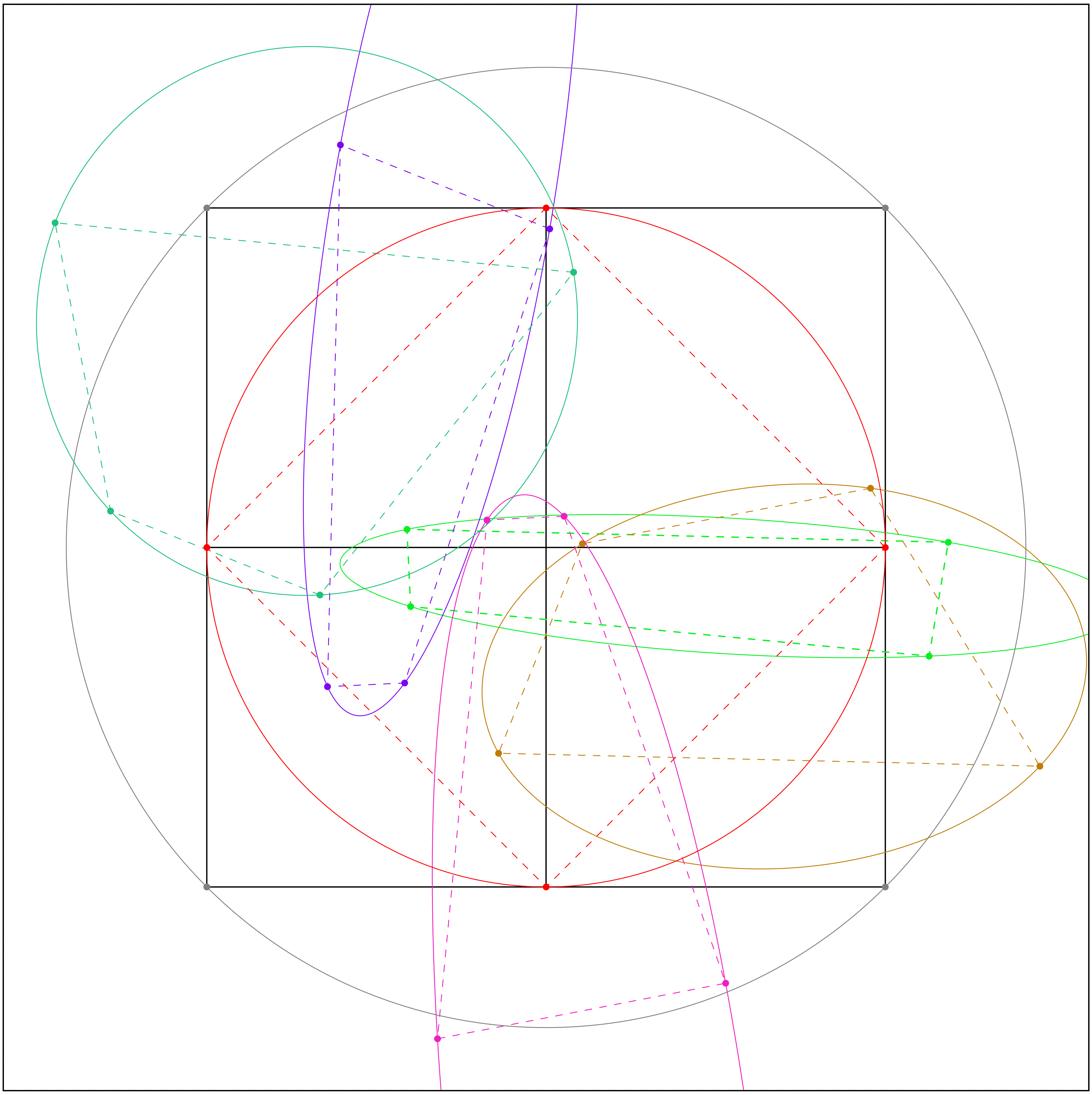
$$\mathbf{p} = (a_0, a_x, a_y, b_0, b_x, b_y, c_x, c_y)^T$$
$$\mathbf{r} = (u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4)^T$$

$$\mathbf{Ap} = \mathbf{r}$$

$$\mathbf{A} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 & -u_1x_1 & -u_1y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 & -u_2x_2 & -u_2y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 & -u_3x_3 & -u_3y_3 \\ 1 & x_4 & y_4 & 0 & 0 & 0 & -u_4x_4 & -u_4y_4 \\ 0 & 0 & 0 & 1 & x_1 & y_1 & -v_1x_1 & -v_1y_1 \\ 0 & 0 & 0 & 1 & x_2 & y_2 & -v_2x_2 & -v_2y_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 & -v_3x_3 & -v_3y_3 \\ 0 & 0 & 0 & 1 & x_4 & y_4 & -v_4x_4 & -v_4y_4 \end{bmatrix}$$

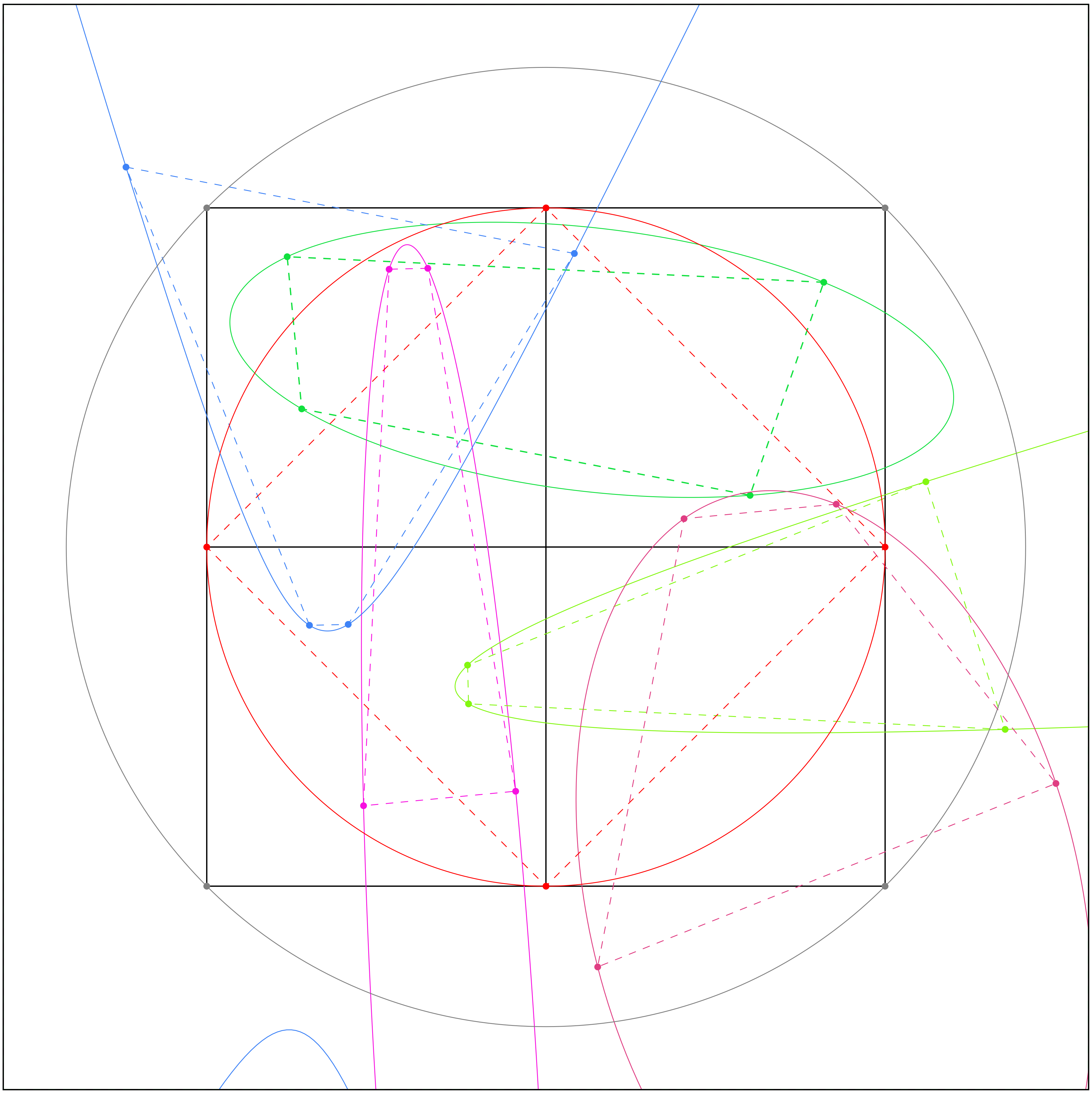
3.1 Examples / Regular cases

The examples were programmed by PostScript.



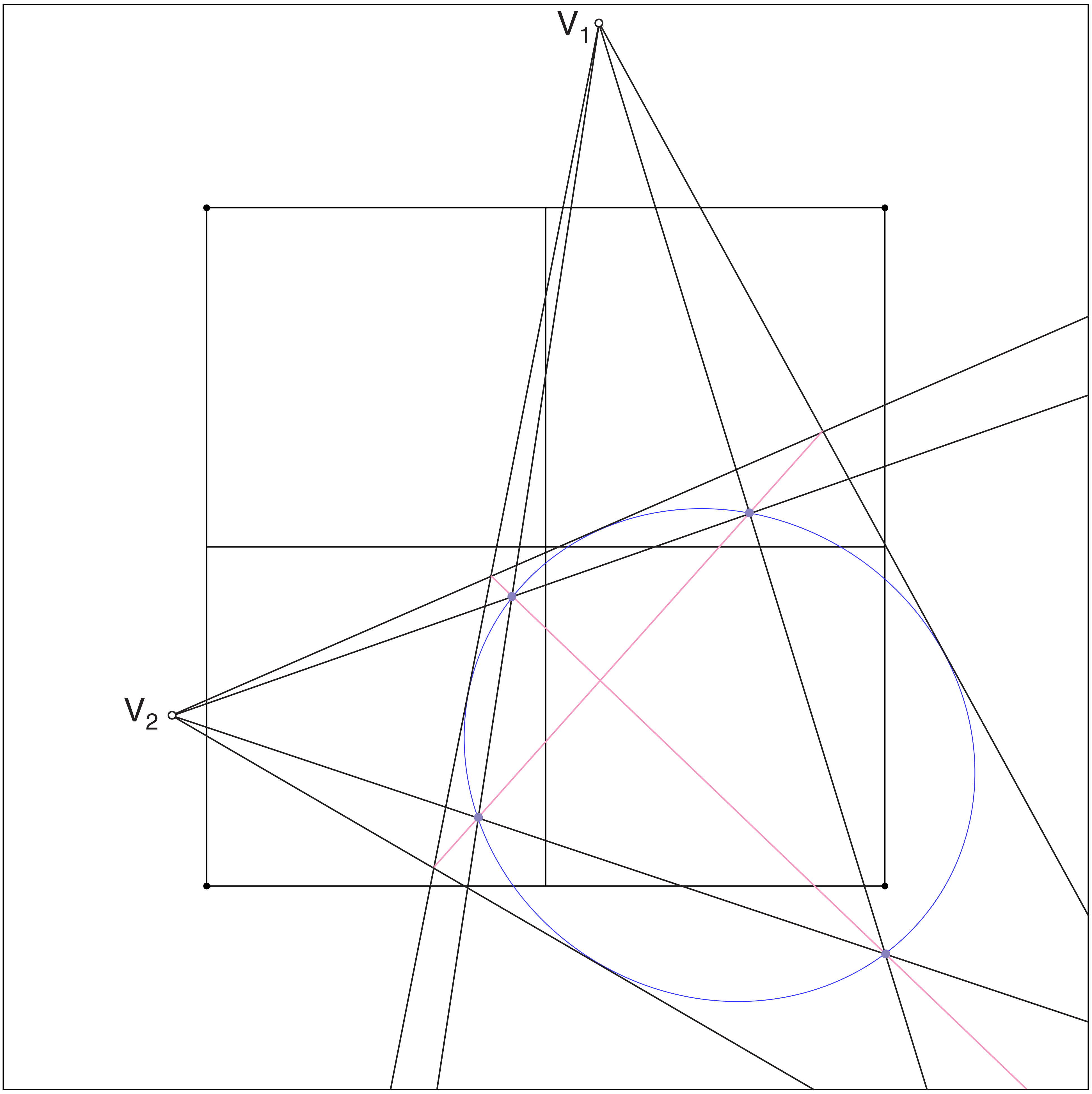
3.2 Examples / Some irregular cases

Two curves are hyperbolas. The blue hyperbola shows something like asymptotes, which is merely a PostScript effect. The green hyperbola doesn´t show the asymptotes in Acrobat Reader.
At present it is not clear in which cases we will get a hyperbola instead of an ellipse. Perhaps so: if the vanishing point would be inside the ellipse then it is a hyperbola (next page).



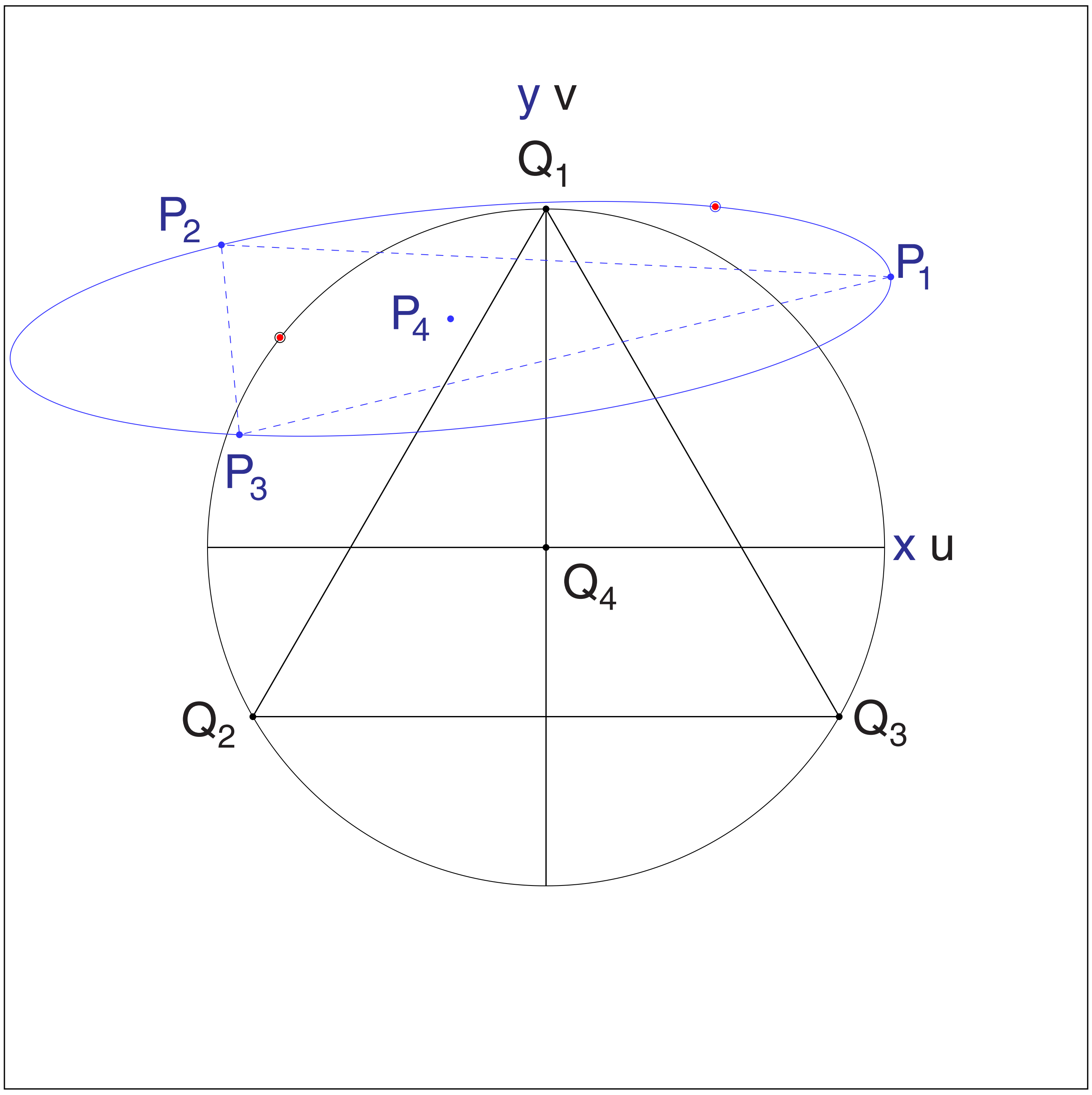
3.3 Examples / Vanishing points

The given points define the vanishing points as well.



4.1 Ellipse through three points

Given are three points $P_1..P_3$ in xy-coordinates (blue) as corners of a triangle. The centroid P_4 is calculated by the mean value of the corner coordinates.
An ellipse should be drawn about P_4 through $P_1..P_3$, the so-called Steiner-Ellipse.
An ellipse is defined by five parameters (center x_0,y_0, half axes a,b and the rotation angle), therefore four points are not sufficient.
The problem is solved by perspective mapping: the ellipse in xy is considered as the perspective image of a circle in uv-coordinates. $Q_1..Q_3$ are the corners of a regular triangle.
Once the functions x(u,v) and y(u,v) are found, the ellipse in xy can be drawn by mapping a sufficiently large number of points from the circle in uv to xy.
The question 'is an arbitrary point inside the ellipse' can be solved by mapping the point to uv, followed by a simple test for 'point in unit circle'. Here we need the inverse transform as described in chapter 2.2. The red point is mapped from the xy space to the uv space.



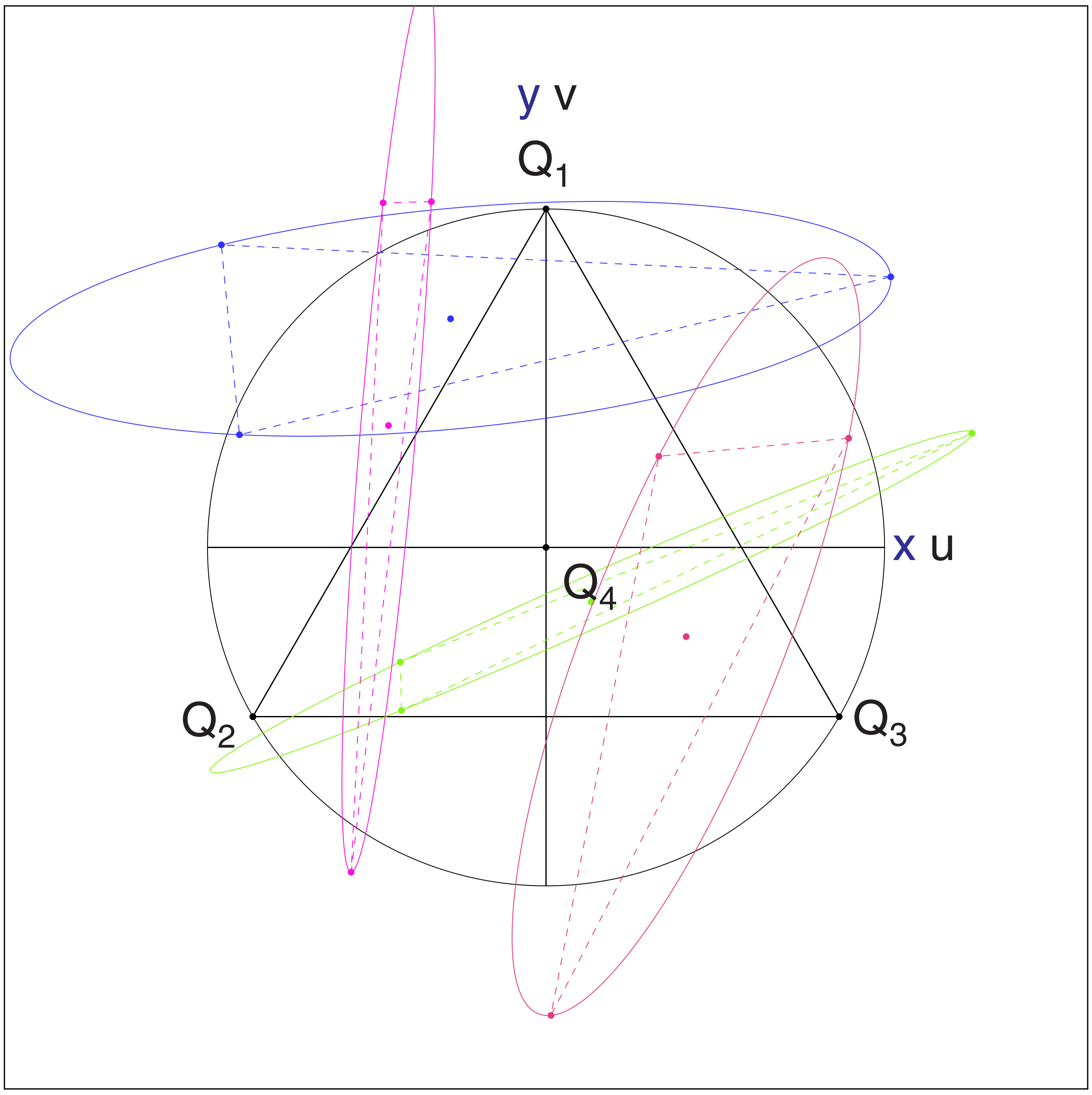
4.2 Ellipse through three points / Mathematics

The mathematics are almost the same as in chapter 2.1:

$$
\begin{aligned}
u_1 &= 0 & v_1 &= 1 \\
u_2 &= -\sqrt{3}/2 & v_2 &= -1/2 \\
u_3 &= +\sqrt{3}/2 & v_3 &= -1/2 \\
u_4 &= 0 & v_4 &= 0
\end{aligned}
$$

$$
\mathbf{A} = \begin{bmatrix}
1 & u_1 & v_1 & 0 & 0 & 0 & -x_1u_1 & -x_1v_1 \\
1 & u_2 & v_2 & 0 & 0 & 0 & -x_2u_2 & -x_2v_2 \\
1 & u_3 & v_3 & 0 & 0 & 0 & -x_3u_3 & -x_3v_3 \\
1 & u_4 & v_4 & 0 & 0 & 0 & -x_4u_4 & -x_4v_4 \\
0 & 0 & 0 & 1 & u_1 & v_1 & -y_1u_1 & -y_1v_1 \\
0 & 0 & 0 & 1 & u_2 & v_2 & -y_2u_2 & -y_2v_2 \\
0 & 0 & 0 & 1 & u_3 & v_3 & -y_3u_3 & -y_3v_3 \\
0 & 0 & 0 & 1 & u_4 & v_4 & -y_4u_4 & -y_4v_4
\end{bmatrix}
$$

4.3 Ellipse through three points / Some examples



5. References

More References are in [1]

[1] G.Hoffmann
Planar Projections
http://www.fho-emden.de/~hoffmann/project18032004.pdf

[2] G.Hoffmann
Rectification by Photogrammetry
http://www.fho-emden.de/~hoffmann/sans04012001.pdf

[3] G.Hoffmann
Perspective Rectification for Images
http://www.fho-emden.de/~hoffmann/persprect13052005.pdf

This doc:
Ellipse through four points
http://www.fho-emden.de/~hoffmann/ellipse08032004.pdf

Last correction January 11 / 2011
p.2: *quadrilateral* instead of *quadriliteral*

Gernot Hoffmann
July 15 / 2006
Website
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