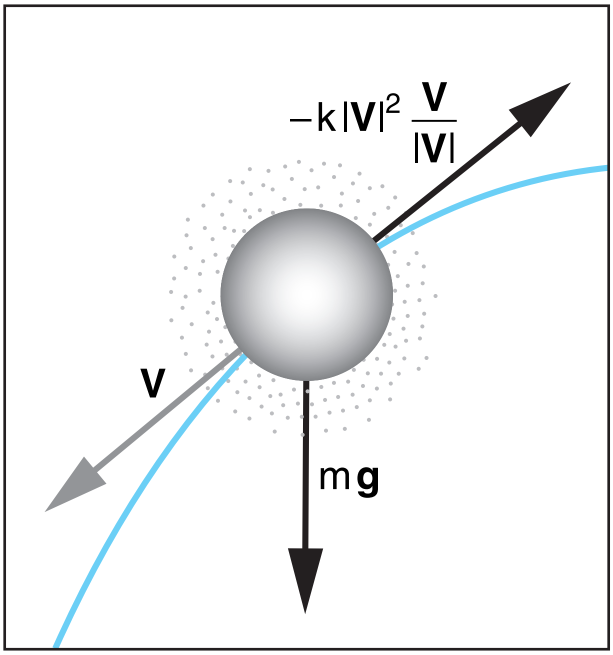
Gernot Hoffmann

Vertical motion with quadratic drag



Contents

Settings for Acrobat (for this doc not very relevant)

Edit / Preferences / General / Page Display (since version 6)
Custom Resolution 72 dpi / View by zoom 100% or 200%

Edit / Preferences / General / Color Management (full version only)
sRGB
Euroscale Coated or ISO Coated or SWOP
Gray Gamma 2.2

1. Introduction

A particle of mass m moves vertically in the field of gravitation. The drag is assumed to be proportional to the square of the speed. The coordinate z and the velocity are counting positive downwards.
The initial velocity V_o=V(0) can have any sign. The final velocity (end velocity) V_e is always positive.
A more general description for the motion of a particle is shown in [4]. The title graphic is also somewhat generalized.

$$m\ddot{z} = mg - kV^2 \text{sign}(V)$$

$$\dot{V} = g(1 - \frac{k}{mg} V^2 \text{sign}(V))$$

$$\dot{z} = V$$

$$V_e = \sqrt{\frac{mg}{k}}$$

2.1 Operation case A

The initial velocity points downwards and is smaller than the end velocity. The solution of the differential equation for the velocity is based on the ‚separation of variables' according to [1].

$$0 \le V < V_e$$

$$0 \le V_o < V_e$$

$$\dot{V} = g(1 - \frac{k}{mg} V^2)$$

Substitution:

$$u = \sqrt{\frac{k}{mg}} V$$

$$\frac{du}{dV} = \sqrt{\frac{k}{mg}}$$

$$dV = \sqrt{\frac{mg}{k}} du$$

$$\sqrt{\frac{m}{gk}} \frac{du}{dt} = 1 - u^2$$

$$\sqrt{\frac{m}{gk}} \frac{du}{1-u^2} = dt$$

Integration ($0 \le u < 1$ or $0 \le V < V_e$):

$$\sqrt{\frac{m}{gk}} \text{artanh}(u) = t + C$$

$$\sqrt{\frac{m}{gk}} \text{artanh}(\sqrt{\frac{k}{mg}} V) = t + C$$

New integration constant c:

$$\text{artanh}(\sqrt{\frac{k}{mg}} V) = \sqrt{\frac{gk}{m}} (t + C) = \sqrt{\frac{gk}{m}} t + c$$

$$V = \sqrt{\frac{mg}{k}} \tanh(\sqrt{\frac{gk}{m}} t + c)$$

2.1 Operation case A continued

With $a = V_e = \sqrt{\frac{mg}{k}}$ and $b = \sqrt{\frac{gk}{m}}$:

$V = a \cdot \tanh(bt + c)$

Transformed:

$V = V_e \tanh(bt + c)$

$$V = V_e \frac{e^{(bt+c)} - e^{-(bt+c)}}{e^{(bt+c)} + e^{-(bt+c)}} = V_e \frac{1 - e^{-2(bt+c)}}{1 + e^{-2(bt+c)}}$$

Calculation of the integration constant:

$V(0) = V_o = V_e \tanh(c)$

$c = \text{artanh}(\frac{V_o}{V_e})$

Transformed:

$$c = 0.5 \ln \frac{1 + V_o/V_e}{1 - V_o/V_e}$$

2.2 Operation case B

The initial velocity points downwards and is larger than the end velocity or equal. The differential equation remains the same but the solution is different.

$V \geq V_e > 0$

$V_o \geq V_e > 0$

$\dot{V} = g(1 - \frac{k}{mg} V^2)$

Substitution as before.

Integration ($u \geq 1$ or $V_o \geq V_e > 0$):

$$V = \sqrt{\frac{mg}{k}} \coth(\sqrt{\frac{gk}{m}}\, t + c)$$

With $a = V_e = \sqrt{\frac{mg}{k}}$ and $b = \sqrt{\frac{gk}{m}}$:

$V = a \cdot \coth(bt + c)$

Transformation:

$V = V_e \coth(bt + c)$

$$V = V_e \frac{e^{(bt+c)} + e^{-(bt+c)}}{e^{(bt+c)} - e^{-(bt+c)}} = V_e \frac{1 + e^{-2(bt+c)}}{1 - e^{-2(bt+c)}}$$

Calculation of the integration constant:

$V(0) = V_o = V_e \coth(c)$

$c = \text{arcoth}(\frac{V_o}{V_e})$

Transformation:

$$c = 0.5 \ln \frac{V_o/V_e + 1}{V_o/V_e - 1}$$

2.3 Operation case C

The initial velocity points upwards. The particle moves upwards until the velocity is zero at the time t_1.
For this case the sign in the differential equation changes from minus to plus.
After t_1 we have case A with $V(t_1)=0$.

$$V \le 0$$
$$V_o < 0$$
$$\dot{V} = g(1+\frac{k}{mg}V^2)$$

Substitution as before.

$$\sqrt{\frac{m}{gk}}\frac{du}{dt} = 1+u^2$$
$$\sqrt{\frac{m}{gk}}\frac{du}{1+u^2} = dt$$

Integration $(u<0 \text{ or } V<0)$:

$$\sqrt{\frac{m}{gk}}\arctan(\sqrt{\frac{k}{mg}}V) = t+C$$

New integration constant c:

$$\arctan(\sqrt{\frac{k}{mg}}V) = \sqrt{\frac{gk}{m}}(t+C) = \sqrt{\frac{gk}{m}}t+c$$
$$V = \sqrt{\frac{mg}{k}}\tan(\sqrt{\frac{gk}{m}}t+c)$$

With $a = V_e = \sqrt{\frac{mg}{k}}$ and $b = \sqrt{\frac{gk}{m}}$:

$$V = a\cdot\tan(bt+c)$$

Calculation of the integration constant:

$$c = \arctan(\frac{V_o}{V_e})$$

Using four-quadrant atan2:

$$c = \text{atan2}(V_o, V_e)$$

The angel is expected in the range 0 to $-\pi/2$.

Calculation of the switching time t_1:

$$0 = \tan(bt_1+c)$$
$$t_1 = -\frac{c}{b}$$

Above t_1 this function has to be used:

Case A with $c=0$

$$V = a\cdot\tanh(b(t-t_1))$$

3. Numerical integration

For the numerical integration the method ‘False Euler’ as described in [3] is sufficient. The standard Euler integration uses on the right side consequently old values. False Euler uses old values in the equation for the velocity but the new value of the velocity in the equation for the coordinate. This is always possible for mechanical systems and delivers more accurate solutions.

$$\dot{V} = g \mp \frac{k}{m} V^2$$

$$\dot{z} = V$$

Initial values:

$$V = V(0)$$

$$z = z(0)$$

Recursion:

$$V := V + dt\,[g \mp \frac{k}{m} V^2]$$

$$z := z + dt\,[V]$$

The sign depends on the case:

Case A: minus

Case B: minus

Case C: up to t_1 plus, then minus

For the numerical integration we do not use t_1 explicitly. The sign in the differential equation changes together with the sign of the velocity.

The time increment is dt=0.03 s (hundred steps for three seconds).

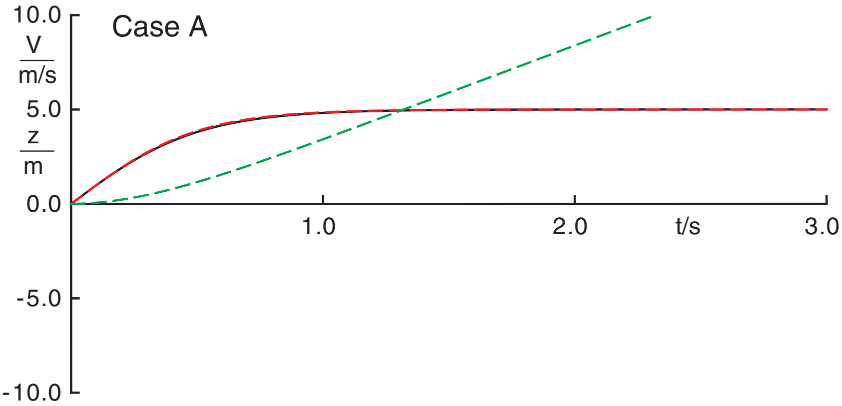
4.1 Examples / Parameters

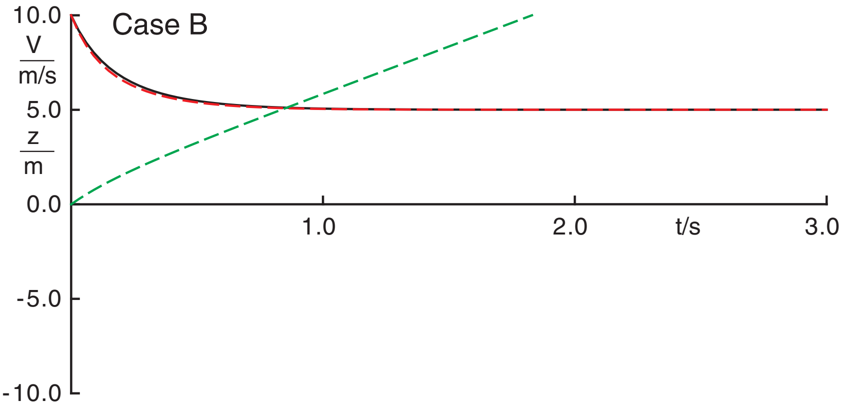
m	=	100	kg
k	=	40	$N/(m/s)^2$
a	=	5	m/s
b	=	2	1/s
g	=	10	m/s^2

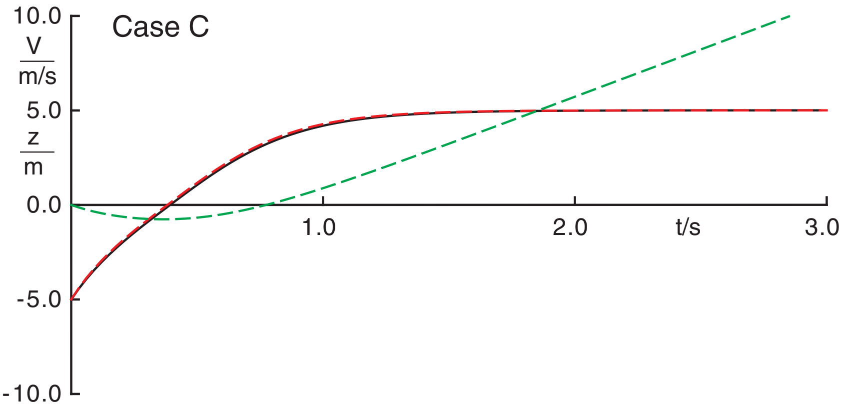
This example is approximately valid for a parachute jumper, operating case A.

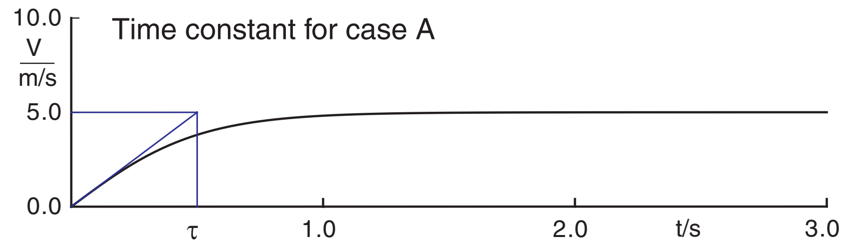
4.2 Examples / Results

Analytical solutions for the velocity are shown black.
Numerical solutions for the velocity are shown red and dashed.
Numerical solutions for the coordinate are shown green and dashed.









The time constant, valid for case A, is calculated by $\tau = 1/b = 0.5\,s$. We have $V(\tau) = 0.7616 V_e$.

5. References

[1] L. Papula
Mathematik für Ingenieure und Naturwissenschaftler, Band 2
Viewegs Fachbücher der Technik
Braunschweig 1994

[2] H.-J.Bartsch
Taschenbuch mathematischer Formeln
Fachbuchverlag Leipzig
1998

[3] G.Hoffmann
Simple integration algorithms: Euler, Verlet, Heun
http://www.fho-emden.de/~hoffmann/integ03082002.pdf
2002

[4] G.Hoffmann
Motion of a masspoint in a landscape
http://www.fho-emden.de/~hoffmann/masspoint09092002.pdf
2002

This document:

http://www.fho-emden.de/~hoffmann/quaddrag01092007.pdf

Gernot Hoffmann
September 04-1 / 2007
Website
Load browser / click here